LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – MATHEMATICS

SECOND SEMESTER – APRIL 2014

MT 2814 - COMPLEX ANALYSIS

Date : 05/04/2014 Dept. No. Max. : 100 Marks
Time : 09:00-12:00

Answer all the questions:

1. (a) Let $\varphi: [a,b] \times [c,d] \to \mathbb{C}$ be a continuous function and define $g: [c,d] \to \mathbb{C}$ by $g(t) = \int_a^b \varphi(s,t)ds$ then prove that g is continuous. Moreover, if $\frac{\partial \varphi}{\partial t}$ exists and is a continuous function on $[a,b] \times [c,d]$ then prove that g is continuously differentiable and $g'(t) = \int_a^b \frac{\partial \varphi(s,t)}{\partial t} ds$. **(5)**

OR

(b) Let G be a connected open set and let $f: G \to \mathbb{C}$ be an analytic function. Then prove that $f \equiv 0$ if and only if there is a point a in G such that $f^{(n)}(a) = 0$ for each $n \geq 0$. **(5)**

(c) (i) Let f be analytic in $B(a;R)$ then prove that $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ for $|z-a| < R$ where $a_n = \frac{1}{n!} f^{(n)}(a)$ and this series has radius of convergence $\geq R$.

(ii) Let G be an open subset of the plane and $f: G \to \mathbb{C}$ an analytic function. If γ is a closed rectifiable curve in G such that $n(\gamma; w) = 0$ for all w in $\mathbb{C} - G$, then prove that for a in $G - \{\gamma\}$, $n(\gamma; a) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} dz$. **(8+7)**

OR

(d) State and prove homotopic version of Cauchy's theorem. **(15)**

2. (a) State and prove Schwarz's lemma. **(5)**

OR

(b) Define convex function and prove that a function $f: [a,b] \to \mathbb{R}$ is convex if and only if $A = \{(x,y): a \leq x \leq b \text{ and } f(x) \leq y\}$ is convex. **(5)**

(c) Let $a < b$ and let G be a vertical strip $\{x + iy: a < x < b\}$. Suppose $f: G^- - \mathbb{C}$ is continuous and f is analytic in G. If we define $M: [a,b] \to \mathbb{R}$ by $M(x) = sup\{|f(x+iy)|\}$, where $-\infty < y < \infty$ and $|f(z)| < B$ for all z in G, then prove that $log M(x)$ is a convex function.

(15)

OR

(d) State and prove Arzela Ascoli theorem. **(15)**

3. (a) If $Re(z_n) > -1$, then prove that $\sum \log(1 + z_n)$ converges absolutely if and only if $\sum z_n$ converges absolutely. **(5)**

OR

(b) With usual notation, Show that $\gamma = \lim_{n\to\infty}\left[\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)-\log n\right]$. **(5)**

(c) (i) State and prove Bohr- Mollerup theorem.

(ii) Show that $\sin\pi z = \pi z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$. **(8+7)**

OR

(d)(i) For $Re\ z > 1$ then prove that $\zeta(z)\Gamma(z) = \int_0^{\infty}(e^t-1)^{-1}t^{z-1}dt$.

(ii) State and prove Euler's theorem. **(7+8)**

4. (a) State and prove Poisson Jensen's formula. **(5)**

OR

(b) Let *f* be an entire function of finite order, then prove that *f* assumes each complex number with one possible exception. **(5)**

(c) State and prove Mittag- Leffler's theorem. **(15)**

OR

(d) State and prove Hadamard factorization theorem. **(15)**

5. (a) If $\wp(z) = \frac{1}{z^2}+\sum_{w\neq 0}\left(\frac{1}{(z-w)^2}-\frac{1}{w^2}\right)$, prove that the series in it is convergent. **(5)**

OR

(b) Show that any elliptic function with periods w_1 and w_2 can be written as $f(z) = c\prod_{k=1}^{n}\frac{\sigma(z-a_k)}{\sigma(z-b_k)}$.

(5)

(c) (i)Define an elliptic function and prove that the sum of the residues of an elliptic function is zero.

(ii) Show that $\zeta(z) = \frac{1}{z}+\sum_{w\neq 0}\left(\frac{1}{(z-w)}+\frac{z}{w^2}+\frac{1}{w}\right)$ and it is an odd function. Also show that $\zeta'(z) = -\wp(z)$. **(7+8)**

OR

(d) (i) State and prove Legendre's relation.

(ii) Prove that $\frac{\wp'(z)}{\wp(z)-\wp(u)} = \zeta(z-u)+\zeta(z+u)-2\zeta(z)$. **(8+7)**