LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

FOURTH SEMESTER – **APRIL 2016**

MT 4810 - FUNCTIONAL ANALYSIS

Date: 15-04-2016 Dept. No. Max. : 100 Marks
Time: 09:00-12:00

Answer all Questions. All questions carry equal marks.

1. (a) Prove that every vector space X contains a set of linearly independent elements that generates X.

(OR)

(b) If $f \in X^*$, then prove that the null space of f, $Z(f)$ has deficiency 0 or 1. **(5 marks)**

(c) Let X be a real vector space, let Y be a subspace of X and p be a real valued function on X such that $p(x+y) \leq p(x)+p(y)$ and $p(ax)=ap(x)$ for all $x, y \in X$, for $a \geq 0$. If f is a linear functional on Y and $f(x) \leq p(x)$ for all $x \in Y$, prove that there is a linear functional F on X such that $F(x)=f(x)$ for all $x \in Y$ and $F(x) \leq p(x)$ for all $x \in X$.

(15 marks)

(OR)

(d) i) If X is a vector space, Y and Z are subspaces of X and is complementary to Z, then prove that every element of X/Y contains exactly one element of Z.

ii) Prove that there is a natural isomorphism between a subspace of X^{**} and X itself.

(6+9 marks)

2. (a) Let X and Y be normed linear spaces and T be a linear transformation. Prove that T is continuous if T is bounded. Is the converse true? Justify.

(OR)

(b) State and prove F.Riesz lemma. **(5 marks)**

(c) State and prove Hahn Banach Theorem for a complex normed linear space.

(OR)

(d) State and prove uniform boundedness principle theorem. **(15 marks)**

3. (a) Let X be a normed vector space and let X' be the dual of X. If X' is separable then prove that X is separable.

(OR)

(b) Let X be a reflexive normed linear space. Prove that every closed subspace of X is reflexive. **(5 marks)**

(c) State and prove open mapping theorem. **(15 marks)**

(OR)

(d) (i) If P is a projection on a Banach space X and if M and N are its range and null space respectively then prove that M and N are closed linear subspaces of X where $X = M \oplus N$.

(ii) If M is a direct sum of X and N is a closed subspace with $X = M \oplus N$ then prove that P is a projection where $Px = y$ for $x = y + z$, $y \in M$, $z \in N$. **(8+7 marks)**

4. (a) State and prove Bessel's inequality.

(OR)

(b) If T is an operator in a Hilbert space X, then show that $(Tx,x) = 0 \square T = 0$. **(5 marks)**

(c) i) Prove that a real Banach space is a Hilbert space if and only if the parallelogram law holds in it.

ii) If T is an operator on a Hilbert space X, show that T is a normal if and only if its real and imaginary parts commute. **(9+6 marks)**

(OR)

(d) If M is a closed subspace of a Hilbert space X, then prove that every $x \in X$ has unique representation $x = y + z,\ y \in M, z \in M^{\perp}$. **(15 marks)**

5. (a) State and prove Riesz-Representation theorem. **(5 marks)**

(OR)

(b) Prove that every zero divisor in Banach algebra A is a topological divisor in A.

(c) State and prove the Spectral theorem.

(OR)

(d) Prove that the mapping $f: G \to G$ given by $f(x) = x^{-1}$ is continuous and a homeomorphism. **(15 marks)**

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