LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION - **MATHEMATICS**

SECOND SEMESTER – **APRIL 2017**

16PMT2MC01/MT 2810 - ALGEBRA

Date : 19-04-2017 Dept. No. Max. : 100 Marks
Time : 01:00-04:00

Answer **ALL** the questions

1. a) If G is a finite group, then prove that $C_a = \dfrac{O(G)}{O\left(N(a)\right)}$. In other words, show that the number of elements conjugate to 'a' in G is the index of the normalizes of 'a' in G.

(OR) **(5)**

b) If p is a prime number and p divides O(G) then G has an element of order p.

c) If p is a prime number and p^{α} divides O(G) then G has a subgroup of order p^{α}.

(OR) (15)

d) Prove that any group of order $11^2 \cdot 13^2$ is abelian and a group of order 72 is not a simple group.

2. a) Given two polynomials $f(x)$, $g(x) \neq 0$ in F[x] then there exists two polynomials t(x), r(x) in F[x] such that $f(x) = t(x)g(x) + r(x)$ where $r(x) = 0$ (or) deg $r(x) < \deg g(x)$.

(OR) **(5)**

b) If $f(x)$ and $g(x)$ are primitive polynomials then $f(x)\,g(x)$ is also a primitive polynomial.

c) (i) Prove x^2+1 is irreducible over the integers module 7. **(7)**
(ii) If $f(x)$ and $g(x)$ are two nonzero polynomials then
$\deg\left(f(x)g(x)\right) = \deg f(x) + \deg g(x)$. (8)

(OR)

d) State and Prove Eisenstein Criterion. **(8)**

e) State and prove Gauss Lemma. **(7)**

3. a) If L is a finite extension of K and K is a finite extension of F then prove that L is a finite extension of F.

(5)

(OR)

b) If $p(x)$ is a polynomial in F[x] of degree $n \geq 1$ and is irreducible over F, then prove that there is an extension of E of F such that $[E : F] = n$ in which $p(x)$ has a root.

c) Prove that the element $a \in K$ is algebraic over F iff F(a) is a finite extension of F.

(15)

(OR)

d) i) If a, b $\in$ K are algebraic over F then show that $a \pm b$, ab and a/b (b$\neq$ o) are algebraic over F. **(8)**

(ii) If F is of characteristic 0 and a, b are algebraic over F, then show that there exists an element $c \in F(a, b)$ such that F (a, b) = F(c). **(7)**

4. a) Prove that K is the normal extension of F iff K is the splitting field of some polynomial over F. **(5)**

(OR)

b) Prove that S_n is not solvable for $n \geq 5$.

c) State and prove the fundamental theorem of Galois Theory.

(OR)

(15)

d) Let K be the normal extension of F and $H \subseteq G(K,F)$, $K_H = \{x \in K / \sigma(x) = x \; \forall \sigma \in H\}$ is the fixed field of the H then prove that

(i) $[K:K_H] = O(H)$ (ii) $H = G(K, K_H)$.

In particular, $H = G(K,F)$ and $[K:F] = O(G(K,F))$.

5. a) For every prime number p and for every positive integer m, prove that there is a unique field having p^m elements.

(OR)

(5)

b) Let G be a finite abelian group such that the relation $x^n = (e)$ is satisfied by atmost n elements of G for every positive integer n then prove that G is a cyclic group.

(c) State and prove Wedderburn's Theorem. **(15)**

(OR)

(d) (i) Let Q be the field of rationals then show that $Q(\sqrt{2}, \sqrt{3}) = Q(\sqrt{2} + \sqrt{3})$. **(8)**

(ii) Let $f(x) = x^2 + 3$ and $g(x) = x^2 + x + 1$ be polynomials over Q. Prove that their splitting fields are equal and find its degree over Q. **(7)**

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