LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

SECONDSEMESTER – APRIL 2017

MT 2814- COMPLEX ANALYSIS

Date: 26-04-2017 Dept. No. Max. : 100 Marks
01:00-04:00

Answer all the questions:

1. (a) Let f be analytic in the disk $B(a;R)$ and suppose that γ is a closed rectifiable curve in $B(a;R)$. Then prove that $\int_\gamma f = 0$. **(5)**

OR

(b) Let G be a connected open set and let $f: G \to \mathbb{C}$ be an analytic function. Then prove that $f \equiv 0$ if and only if there is a point a in G such that $f^{(n)}(a) = 0$ for each $n \geq 0$.

(5)

(c) (i) Let $f: G \to \mathbb{C}$ be analytic and suppose $\bar{B}(a;r) \subset G (r > 0)$. If $\gamma(t) = a + re^{it}$, $0 \leq t \leq 2\pi$, then prove that $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw$ for $|z-a| < r$.

(ii) State and prove Cauchy's Estimate. **(10+5)**

OR

(d) (i) If $\gamma: [0,1] \to \mathbb{C}$ is a closed rectifiable curve and $a \notin \{\gamma\}$ then prove that $\frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}dz$ is an integer.

(ii) Let G be an open set and let $f: G \to \mathbb{C}$ be a differentiable function, then prove that f is analytic on G. **(5+10)**

2. (a) Let $0 < R_1 < R_2 < \infty$ and suppose f is analytic on $ann(0;\ R_1,R_2)$. If $R_1 < r < R_2$, define $M(r) = max\{|f(re^{i\theta})|: 0 < \theta < 2\pi\}$ then prove that $logM(r)$ is a convex function of $logr$.

(5)

OR

(b) Prove that a differentiable function f on $[a,b]$ is convex if and only if f' is increasing. **(5)**

(c) State and prove Arzela Ascoli theorem. **(15)**

OR

(d) State and prove Riemann Mapping theorem. **(15)**

3. (a) Let X be a set and let $f, f_1, f_2, \ldots$ be functions from X into $\mathbb{C}$ such that $f_n(x) \to f(x)$ uniformly for $x \in X$. If there is a constant a such that $Re\, f(x) \leq a$ for all $x \in X$ then prove that $\exp f_n(x) \to \exp f(x)$ uniformly for $x \in X$. **(5)**

OR

(b) Let $Re z_n > 0$, for all $n \geq 1$. Then prove that $\prod_{k=1}^{\infty} z_k$ converges to a complex number different from zero if and only if $\sum_{k=1}^{\infty} log z_k$ converges. **(5)**

(c) (i) Let (X, d) be a compact metric space and let $\{g_n\}$ be a sequence of continuous functions from X into $\mathbb{C}$ such that $\sum g_n(x)$ converges absolutely and uniformly for x in X. Then prove that the product $f(x) = \prod_{n=1}^{\infty}(1 + g_n(x))$ converges absolutely and uniformly for x in X. Also prove that there is an integer n_0 such that $f(x) = 0$ if and only if $g_n(x) = -1$ for some n, $1 \leq n \leq n_0$.

(ii) Prove that (a) $\left\{\left(1 + \frac{z}{n}\right)^n\right\}$ converges to e^z in $H(\mathbb{C})$ (b) If $t \geq 0$ then $\left(1 - \frac{t}{n}\right)^n \leq e^{-t}$ for all $n \geq t$.

(8+7)

OR

(d) State and prove Weierstrass factorization theorem. **(15)**

4. (a) Let f be an entire function of finite order, then prove that f assumes each complex number with one possible exception. **(5)**

OR

(b) State and prove Jensen's formula. **(5)**

(c) State and prove Mittag- Leffler's theorem. **(15)**

OR

(d) Let f be a non-constant entire function of order λ with $f(0) = 1$, and let $\{a_1, a_2, \ldots\}$ be the zeros of f counted according to multiplicity and arranged so that $|a_1| \leq |a_2| \leq \cdots$. If an integer $p > \lambda - 1$ then prove that $\frac{d^p}{dz^p}\left(\frac{f'(z)}{f(z)}\right) = -p!\, \sum_{n=1}^{\infty} \frac{1}{(a_n - z)^{p+1}}$ for $z \neq a_1, a_2, \ldots$

(15)

5. (a) Prove that any two bases of a same module are connected by a unimodular transformation.

(5)

OR

(b) Define $\zeta(z)$ and derive the relation between $\zeta(z)$ and $\sigma(z)$. **(5)**

(c) Prove that $\wp(z)$ is an elliptic function. **(15)**

OR

(d) (i) Show that $\begin{vmatrix} \wp(z) & \wp'(z) & 1 \\ \wp(u) & \wp'(u) & 1 \\ \wp(u+z) & -\wp'(u+z) & 1 \end{vmatrix} = 0.$

(ii) State and prove Legendre's relation. **(7+8)**

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