LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc.DEGREE EXAMINATION – **MATHEMATICS**

SECONDSEMESTER – APRIL 2018

17/16PMT2MC02- MEASURE THEORY AND INTEGRATION

Date: 19-04-2018 Dept. No. Max. : 100 Marks

Time: 01:00-04:00

Answer ALL the questions. Each question carries equal marks.

1. a) Let E_1, E_2 be two measurable sets with $E_1 \supseteq E_2$. Prove that $E_1 - E_2$ is measurable and its measure is $m(E_1) - m(E_2)$. (5)

OR

b) Show that if *f* is measurable, then $\{x: f(x) = \alpha\}$ is measurable for each extended real number α. (5)

c) (i) Construct a non-measurable set.

(ii) If $F \in \mathcal{M}$, the class of Lebesgue measurable setsand $m^*(F \triangle G) = 0$,then prove that $G \in \mathcal{M}$. (11 + 4)

OR

d) Show that every Borel set is measurable but the converse is not true. (15)

2. a) Let *f* be a non-negative measurable function. Prove that there exists a sequence $\{\varphi_n\}$ of measurable simple functions such that for each *x*, $\varphi_n(x) \uparrow f(x)$. (5)

OR

b) Evaluate$\int_0^1 \left(\frac{logx}{1-x}\right)^2 dx$. (5)

c) (i) State and prove Lebesgue Dominated convergence theorem.

(ii) Show that if α> 1, $\int_0^1 \frac{xsinx}{1+(nx)^\alpha} dx = \circ\, (n^{-1}) as\, n \longrightarrow \infty$. (10 + 5)

OR

d) Prove that Riemann integrability implies Lebesgueintegrability. Is the converse true?Justify. (15)

3. a) Show that every algebra is a ring and every σ-algebra is a σ-ring but the converse is not true. (5)

OR

b) Let $[\![X, S, \mu]\!]$ be a measure space and *f*be a non-negative measure function. Prove that $\varphi(E) = \int_E f d\mu$ is a measure on the measurable space $[\![X, S]\!]$. Also if $\int f d\mu < \infty$, prove that for all $\epsilon > 0$ there exists$\delta > 0$ such that $A \in S$ and $\mu(A) < \delta$ then $\varphi(A) < \epsilon$. (5)

c) (i) Let μ^* be the outer measure on $\mathcal{H}(\mathfrak{R})$defined by μ on a ring $\mathfrak{R}$. Prove that the class of μ^*-measurable sets S^* contains the σ-ring generated by $\mathfrak{R}$.

(ii) Ifμ is a σ-finite measure on a ring$\mathfrak{R}$then prove that it has an unique extension to the σ-ring $\mathcal{S}(\mathfrak{R})$. (6 + 9)

OR

d) If μ is a measure on a σ-ring $\mathcal{S}$ then prove that the class $\bar{\mathcal{S}}$ of sets of the form $E\Delta N$ for any sets E, N such that $E \in \mathcal{S}$ while N is contained in some set in $\mathcal{S}$of zero measure is σ-ring. Also prove that the set function $\bar{\mu}$ defined by $\bar{\mu}(E\Delta N) = \mu(E)$ is a complete measure on $\bar{\mathcal{S}}$. (15)

4. a) Prove that every convex function on an open interval is continuous. (5)

OR

b) Let $f_n \to f$ almost uniformly , then prove the following: (i) $f_n \to f$ in measure (ii) $f_n \to f$ in almost everywhere. (3 + 2)

c) (i) State and prove Holder's inequality. When will equality occur?

(ii) Establish the inequality $\left|\int_0^{\pi} x^{-\frac{1}{4}} sinxdx\right| \leq \pi^{3/4}$. (10 + 5)

OR

d) (i) State and prove completeness theorem for convergence in measure.

(ii) Let $\{f_n\}$ be a sequence of non negative measurable functions and let f be a measurable function such that $f_n \to f$ in measure then prove that $\int f d\mu \leq \liminf \int f_n d\mu$. (8 + 7)

5. a) Define a positive and null set with respect to the signed measure ν on $[\![X, S]\!]$ and prove that a countable union of positive sets with respect to a signed measure ν is a positive set.(5)

OR

b) Let μ, λ , ν are σ-finite signedmeasures on $[\![X, S]\!]$ such that, $\nu \ll \mu, \mu \ll \lambda$ then show that $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} . \frac{d\mu}{d\lambda} [\lambda]$. (5)

c) State and prove Hahn decomposition theorem. Give an example showing that a Hahn decomposition is not unique. (15)

OR

d) State and prove Radon-Nikodym theorem. (15)

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