LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc., DEGREE EXAMINATION - MATHEMATICS

SECOND SEMESTER – APRIL 2019

18PMT2MC01/MT 2810 - ALGEBRA

Date: 03-04-2019 Dept. No. Max. : 100 Marks

Time: 01.00 : 04.00

Answer **ALL** the questions

1. a) Prove that a group of order 72 is not simple.

OR **(5)**

b. Let G be a finite group. Then prove that $C_a = O(G)/O(N(a))$ *i.e.,* the number of elements conjugate to *'a'* in G is the index of the normalizer of *'a'* in G. Also prove that a □ Z *iff* $N(a)=G$.

c) If p is a prime number such that p^{α} divides order of G then prove that G has a subgroup of order p^{α}. **(15)**

OR

d) If p is a prime number and p divides $O(G)$ then G has an element of order p. **(8)**

e) Prove that any group of order $11^2.13^2$ is abelian. **(7)**

2. a) State and prove Division Algorithm.

OR **(5)**

b) If $f(x)$ and $g(x)$ are primitive polynomials then prove that $f(x)g(x)$ is also a primitive polynomial.

c) $O(G)=p^n$ where p is the prime number then $Z(G)\neq e$ *i.e.,* $O(G)>1$. In other words a group of prime orders must always have a nontrivial centre. Also prove that if $O(G) = p^2$ where p is a prime number then G is abelian. **(8)**

d) If $f(x)$ and $g(x)$ are two nonzero polynomials, then prove that $deg(f(x)g(x)) = deg(f(x)) + deg(g(x))$. **(7)**

OR

e) State and prove Gauss Lemma.

(8)

f) Let $f(x) = a_0 + a_1x + \ldots + a_nx^n$ be a polynomial with integer coefficients. Suppose for some prime number p, $p \nmid a_n$, $p \mid a_1$, $p \mid a_2, \ldots, p|a_0$, $p^2 \nmid a_0$ then prove that $f(x)$ is irreducible over Q. **(7)**

3. a) The elements in K which are algebraic over F form a subfield of K.

OR **(5)**

b) Let Q be the rationals. Then show that $Q(\sqrt{2}, \sqrt{3}) = Q(\sqrt{2} + \sqrt{3})$.

c) The element a □ K is said to be algebraic over F *iff* $F(a)$ is a finite extension over F. **(15)**

OR

d) If L is the finite extension of K and K is the finite extension of F then prove that L is the finite extension of F. **(10)**

e) If L is the finite extension of F and K is the subfield of L which contains F then prove that $[K:F]$ divides $[L:F]$. **(5)**

4. a) If F is of characteristic 0 and a,b are algebraic over F, then show that there exists an element $c \in F(a,b)$ such that $F(a, b) = F(c)$.

OR **(5)**

b) Find the degree of the splitting field x^3-2 over Q and x^4+x^2+1 over Q.

c) State and prove fundamental theorem of Galois Theory.

OR **(15)**

d) Let K be a normal extension of F and let H be a subgroup *of* $G(K,F)$, $K_H = \{x \in K \,/\, \sigma(x) = x \;\forall\, \sigma \in H\}$ is a fixed field of H then prove that (i) $[K:K_H] = O(H)$, (ii) $H = G(K,K_H)$. In particular, $H = G(K,F)$, $[K:F] = O(G(K,F))$.

5. a) Let G be a finite abelian group. Show that the relation $x^n = (e)$ is satisfied by at most n elements of G for every positive integer n then prove that G is a cyclic group.

OR **(5)**

b) Prove that the polynomial $f(x) \in F[x]$ has multiple root *iff* $f(x) + f'(x)$ have a nontrivial common factor.

c) Prove that any finite division ring is necessarily a commutative field.

OR **(15)**

d) Prove that S_n is not solvable for $n \geq 5$ and verify S_3 is solvable.

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