LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

FOURTH SEMESTER – **APRIL 2022**

PMT 4501 – FUNCTIONAL ANALYSIS

Date: 15-06-2022 Dept. No. Max. : 100 Marks

Time: 01:00 PM - 04:00 PM

Answer all questions. All questions carry equal marks.

1. (a) If X is a vector space, Y and Z are subspaces of X and Y is complementary to Z, prove that every element of X/Y contains exactly one element of Z.

OR

(b) Define Hamel basis. If S is a Hamel basis of X prove that every x belonging to X has unique representation as a linear combination of finite number of elements of S. (5 marks)

(c) Let X be a real vector space, let Y be a subspace of X and p be a real valued function on X such that $p(x+y) \leq p(x) + p(y)$ and $p(ax) = ap(x)$ for all $x, y \in X$, for $a \geq 0$. If f is a linear functional on Y and $f(x) \leq p(x)$ for all $x \in Y$, prove that there is a linear functional F on X such that $F(x) = f(x)$ for all $x \in Y$and $F(x) \leq p(x)$ for all $x \in X$. (15 marks)

OR

(d) (i) Prove that every vector space X contains a set of linearly independent elements that generates X.

(ii) If X is a vector space, Y and Z are subspaces of X, prove that for every $x \in X$ there are elements $y \in Y$ and $z \in Z$ such that $x = y + z$ and this representation is unique. (10+5 marks)

2. (a) Let X and Y be normed linear spaces and T be a linear transformation. Prove that T is continuous if and only if T is bounded.

OR

(b) Let $B(X,Y)$ be the set of all bounded linear transformation of X into Y. Prove that $B(X,Y)$ is a normed vector space which is Banach space if Y is a Banach space. (5 marks)

(c) (i) Let X be a real normed linear space and Y be a subspace of X. Let f be a bounded linear functional on Y with $\|f\|$ relative to Y. Prove that f has a continuous linear extension to an $x' \in X'$ with $\|x'\| = \|f\|$.

(ii) Let X be a real normed linear space. Then prove that for any $x \neq 0$ in X there is an $x' \in X'$ such that $x'(x) = \|x\|$ and $\|x'\| = 1$. (9+6 marks)

OR

(d) State and prove uniform boundedness principle theorem. (15 marks)

3. (a) Let X be a normed vector space and let X' be the dual of X. If X' is separable prove that X is separable.

OR

(b) State and prove the closed graph theorem. (5 marks)

(c) If X and Y are Banach spaces and if T is a continuous linear transformation of X onto Y, prove that T is an open mapping. (15 marks)

OR

(d) (i) Define projection on a Banach space. Mention the necessary and sufficient conditions for a linear space X to be a direct sum of its two subspaces. (3+6+6 marks)

(ii) Define range and nullspace on a Banach space. If P is a projection on a Banach space X and if M and N are its range and null space respectively, prove that M and N are closed linear subspaces of X such that $X = M \oplus N$.

(iii) If M is a direct sum of X and N is a closed subspace with $X = M \oplus N$ and with unique representation $x = y + z$ where $y \in M$, $z \in N$ prove that P is a projection where $Px = y$.

4. (a) State and prove Bessel's inequality.

OR

(b) If T is an operator on a Hilbert space X, show that T is a normal if and only if its real and imaginary parts commute. (5 marks)

(c) Prove that a real Banach space is a Hilbert space if and only if the parallelogram law holds.

OR

(d) If M is a closed subspace of a Hilbert space X, then prove that every $x \in X$ has unique representation $x = y + z,\ y \in M, z \in M^{\perp}$. (15 marks)

5. (a) Define Banach algebra, set of regular elements G and the set of singular elements S. Prove that G is an open set and therefore S is a closed set. (3+2 marks)

OR

(b) If B is a B^* - algebra, prove that $\|x^*\| = \|x\|$ and $\|x^* x\| = \|x^*\|\|x\|$. If x is a normal element in a B^* - algebra prove that $\|x^2\| = \|x\|^2$.

(2+1+2 marks)

(c) State and prove the Spectral theorem.

OR

(d) (i) Let A be Banach algebra. Show that the inverse of the regular element $x \in A$ is $x^{-1} = 1 + \sum_{n=1}^{\infty} (1-x)^n$.

(ii) Prove that the mapping $f : G \to G$ given by $f(x) = x^{-1}$ is continuous and is a homeomorphism. (7+8 marks)

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