LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

FOURTH SEMESTER – **APRIL 2023**

PMT 4501 – FUNCTIONAL ANALYSIS

Date: 29-04-2023 Dept. No. Max. : 100 Marks

Time: 09:00 AM - 12:00 NOON

Answer all questions. All questions carry equal marks.

1. (a) If X is a vector space, Y and Z are subspaces of X, prove that for every $x \in X$ there are elements $y \in Y$ and $z \in Z$ such that $x = y + z$ and this representation is unique.

OR

(b) Define Hamel basis. If X is a vector space, prove that all Hamel bases of X have the same cardinal number. (5 marks)

(c) (i) If X is a vector space, Y and Z are subspaces of X and Y is complementary to Z, prove that every element of X/Y contains exactly one element of Z.

(ii) State Zorn's lemma and prove that every vector space X contains a set of linearly independent elements which generates X. (5+10 marks)

OR

(d) (i) Let X be a vector space and Y be a subspace of X. Define deficiency and hyperplane. Is every proper subspace of a vector space contained in a hyperplane?

(ii) Prove that if $f \in X^*$, then $Z(f)$ has deficiency 0 or 1 in X. Also prove that, if Z is a subspace of X of deficiency 0 or 1, then there is an $f \in X^*$ such that $Z = Z(f)$. (3+12 marks)

2. (a) Let T be a linear transformation. If T is continuous at $x = 0$, prove that it is continuous everywhere and the continuity is uniform. Also prove that T is continuous when bounded.

OR

(b) Let $B(X,Y)$ be the set of all bounded linear transformation of X into Y. Prove that $B(X,Y)$ is a normed vector space which is Banach space if Y is a Banach space. (5 marks)

(c) State and prove Hahn Banach Theorem for a complex normed linear space.

OR

(d) State and prove Banach-Steinhaus theorem. (15 marks)

3. (a) If X is a vector space and X^{**} is a second dual of X, prove that there is a natural isomorphism between certain subspaces X^{**} and X itself.

OR

(b) Explain projection in Banach space with necessary diagram and examples. (5 marks)

(c) State and prove open mapping theorem. (15 marks)

OR

(d) (i) X and Y are Banach spaces and T is a linear transformation of X into Y. Prove that $G(T)$ is closed in $X \times Y$ and $D(T)$ is closed in $X \Rightarrow Y$ is bounded. (5+7+3 marks)

(ii) If X is finite dimensional normed linear space over a field F, prove that $\dim X = \dim X^*$.

(iii) What do you infer from the open mapping theorem and closed graph theorem?

4. (a) Prove the identity which relates the inner product to the norm in Hilbert space. (5 marks)

OR

(b) Let $\{x_1, x_2, \dots, x_n\}$ be an orthonormal set. Prove that for any $x \in H$, $\sum_{i=1}^{n} |\langle x, x_i \rangle|^2 \leq \|x\|^2$.

(c) Prove that a real Banach space is a Hilbert space if and only if the parallelogram law holds.

OR

(d) (i) State and prove Dual Riesz representation theorem. (10+5 marks)

(ii) Prove that an operator T on X is self-adjoint if and only if (Tx, x) is real for all x.

5. (a) Let A be a Banach algebra and $x \in A$. Prove that $\sigma(x)$ the spectrum of x is non-empty.

OR

(b) Define topological divisor of zero. Prove that every zero divisor in Banach algebra A is a topological divisor in A. (5 marks)

(c) (i) Let r be an element of radical R, $x \in A$ and $1 - xr$ is regular. Prove that $1 - rx$ is regular.

(ii) Let A be a Banach algebra and $x \in A$. Prove that the spectral radius is given by $r(x) = \lim \|x^n\|^{1/n}$. (3+12 marks)

OR

(d) State and prove the Spectral theorem. (15 marks)

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