LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

FIRST SEMESTER – **NOVEMBER 2016**

16PMT1MC01/MT 1815 – LINEAR ALGEBRA

Date: 02-11-2016 Dept. No. Max. : 100 Marks

Time: 01:00-04:00

Answer ALL the questions:

I. a. i) Find the minimal polynomial for the matrix

$$A = \begin{pmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{pmatrix}.$$

(OR) **(5)**

ii) Let T be a linear operator on a finite dimensional space V and let c be a scalar. Prove that the following statements are equivalent.

1. c is a characteristic value of T.
2. The operator $(T-cI)$ is singular.
3. det $(T-cI)$ =0.

b. i) Let T be a linear operator on a finite dimensional space V and $c_1, \dots c_k$ be the distinct characteristic values of T. Let W_i be the null space of $(T - c_iI)$. Prove that the following are equivalent.

1. T is diagonalizable
2. The characteristic polynomial for T is $f = (x-c_1)^{d1} \dots (x-c_k)^{d_k}$ and dim $W_i = d_i, i = 1, \dots, k$.
3. dim $W_1 + \dots$ dim W_k = dim V.

(OR) **(15)**

ii) State and prove Cayley – Hamilton Theorem.

II. a. i) Let V be a finite-dimensional vector space. Let $W_1, \dots, W_K$ be subspaces of V and let $W = W_1 + \dots + W_K$. The following are equivalent.

a) $W_1, \dots, W_K$ are independent.

b) For each j, $2 \le i \le k$, we have $Wj \cap (W_1 + \dots + W_{j-1}) = \{0\}$.

(OR) **(5)**

ii) Let W be an invariant subspace for T. Then prove that the minimal polynomial for T_w divides the minimal polynomial for T.

b. i) State and prove Primary Decomposition theorem.

(OR) **(15)**

ii) Let T be a linear operator on a finite dimensional space V. If T is diagonalizable and if $c_1, \dots, c_k$ are the distinct characteristic values of T, then prove that there exist linear operators $E_1, \dots, E_k$ on V such that

1. $T = c_1E_1 + \dots + c_kE_k$ 2. $I = E_1 + \dots + E_k$ 3. $E_iE_j = 0, i \ne j$ 4. Each E_i is a projection

5. The range of E_i is the characteristic space for T associated with c_i.

III. a. i) If $\mathcal{B}$ is an ordered basis for W_i, $1 \le i \le k$, then prove that the sequences $\mathcal{B} = (\mathcal{B}_1 \dots \mathcal{B}_k)$ is an ordered basis for W.

(OR) **(5)**

ii) Let V be a finite-dimensional vector space over the field F and let T be a linear operator on V. Then T is triangulable if and only if the minimal polynomial for T is a product of linear polynomials over F.

b. i) State and prove cyclic decomposition theorem. **(15)**

(OR)

ii) If W is T – admissible then, there exists a vector $\alpha \in \nu$ such that $W \cap Z(\alpha;T)=\{0\}$. **(7)**

iii) Let T be a linear operator on a finite-dimensional vector space V. Let p and f be the minimal and characteristic polynomials for T, respectively.

(1) p divides f.

(2) p and f have the same prime factors, except for multiplicities.

(3) If $p = f_1^{r_1} \ldots f_k^{r_k}$

In the prime factorization of p, then $f = f_1^{d_1} \ldots f_k^{d_k}$ where d_i is the nullity of $f_i\,(T)^{r_i}$ divided by the degree of f_i. **(8)**

IV. a. i) Let V be a complex vector space and f be a form on V such that f(α, α) is real for every α, then f is Hermitian. **(5)**

(OR)

ii) Let *T* be a linear operator on a complex finite dimensional inner product space *V*. Then prove that *T* is self–adjoint if and only if $(T\alpha / \alpha)$ is real for every α in *V*.

b. i) Let V be a finite-dimensional inner product space and f a form on V. Then there is a unique linear operator T on V such that f(α, β) = (Tα|β) for all α, β in V, and the map f→ T is an isomorphism of the space of forms onto L(V, V). **(8)**

ii) For any linear operator T on a finite-dimensional inner product space V, there exists a unique linear T* on V such that (Tα|β) = (α|T*β) for all α, β in V. **(7)**

(OR)

iii) Let F be the field of real numbers or the field of complex numbers. Let A be an n x n matrix over F. Show that the function g defined by g (X, Y) = Y* AX is a positive form on the space F^{nx1} if and only if there exists an invertible n x n matrix P with entries in F such that A = P*P. **(7)**

iv) State and prove Principal Axis Theorem. **(8)**

V. a. i) Define Quadratic form, Bilinear form, symmetric bilinear form and prove that

$$f(\alpha,\beta)=\frac{1}{4}q\,(\alpha+\beta)-\frac{1}{4}q(\alpha-\beta).$$

(OR) **(5)**

ii) Let V be a finite dimensional vector space over a field of characteristic zero and let *f* be symmetric bilinear form on V. Then prove that there is an ordered basis for V in which *f* is represented by a diagonal matrix.

b. i) Let V be an n-dimensional vector space over the field of real numbers, and let f be a symmetric bilinear form on V which has rank r. Then there is and ordered basis $\{\beta_1,\beta_2,\ldots,\beta_n\}$ for V in which the matrix of f is diagonal and such that

$f(\beta_j, \beta_j)=\pm 1, \quad j = 1,\ldots, r$. Also show that the number of basis vectors β_j for which $f(\beta_j,\beta_j) = 1$ is independent of the choice of basis. **(7)**

ii) Let V be a finite-dimensional vector space over the field of complex numbers. Let f be a symmetric bilinear form on V which has rank r. Then prove that there is an ordered basis $\mathcal{B} =\{\beta_1,\ldots,\beta_n\}$ for V such that: (1) the matrix of f in the ordered basis $\mathcal{B}$ is diagonal; **(8)**

(2) $f(\beta_j,\beta_j)=\begin{cases}1, & j=1,\ldots,r\\ 0, & j>r\end{cases}$........

(OR)

iii) Let V and W be the finite dimensional inner product space over the same field having the same dimension. If T is a linear transformation from V→W, then the following are equivalent.

(i) T preserves inner products. (ii) T is an isomorphism.

(iii) T carries every orthonormal basis of V onto an orthonormal basis for W.

(iv) T carries some orthonormal basis of V onto an orthonormal basis for W.

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