LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

SECOND SEMESTER – NOVEMBER 2016

MT 2814 - COMPLEX ANALYSIS

Date: 15-11-2016 Dept. No. Max. : 100 Marks

Time: 01:00-04:00

Answer all the questions.

1. a) State and prove Cauchy's Estimate. **(5)**

OR

b) Define (i) Zeros of an analytic function (ii) index of a closed curve (iii) FEP homotopic (iv) Simply connected. **(5)**

c) State and prove Goursat's theorem. **(15)**

OR

d) State and prove homotopic version of Cauchy's theorem. **(15)**

2. a) State and prove Hadamard's three circles theorem. **(5)**

OR

b) Prove that a differentiable function *f* on $[a, b]$ is convex if and only if f' is increasing. **(5)**

c) Prove that any set $\mathfrak{F} \subset C(G, \Omega)$ is normal if and only if the following conditions are satisfied: (i) for each *z* in *G*, $\{f(z) : f \in \mathfrak{F}\}$ has compact closure in Ω (ii) $\mathfrak{F}$ is equicontinuous at each point of *G*. **(15)**

OR

d) Let G be a region which is not the whole plane and let $a \in G$ then prove that there is a unique analytic function $f: G \to C$ having the properties (a) $f(a) = 0$ and $f'(a) > 0$ (b) f is one-one and (c) $f(G) = D = \{z: |z| < 1\}$. **(15)**

3. a) Show that $\sin \pi z = \pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$. **(5)**

OR

b) If $|z| \leq 1$ and $p \geq 0$ then prove that $\left|1 - E_p(z)\right| \leq |z|^{p+1}$. **(5)**

c) (i) If $Re\ z_n > -1$ then prove that $\sum log(1 + z_n)$ converges absolutely if and only if $\sum z_n$ converges absolutely.

(ii) Let (X, d) be a compact metric space and let $\{g_n\}$ be a sequence of continuous functions from X into $\mathbb{C}$ such that $\sum g_n(x)$ converges absolutely and uniformly for x in X. Then prove that the product $f(x) = \prod_{n=1}^{\infty}(1 + g_n(x))$ converges absolutely and uniformly for x in X. Also prove that there is an integer n_0 such that $f(x) = 0$ if and only if $g_n(x) = -1$ for some n, $1 \leq n \leq n_0$.

(7+8)

OR

d) (i) State and prove Bohr-Mollerup theorem.

(ii) Let X be a set and let $f, f_1, f_2, \ldots$ be functions from X into $\mathbb{C}$ such that $f_n(x) \to f(x)$ uniformly for $x \in X$. If there is a constant a such that $Re\, f(x) \leq a$ for all $x \in X$ then prove that $\exp f_n(x) \to \exp f(x)$ uniformly for $x \in X$. **(10+5)**

4. a) State and prove Jensen's formula. **(5)**

OR

b) If f is an entire function with finite order λ, where λ is not an integer then prove that f has infinitely many zeros. **(5)**

c) Let f be a non-constant entire function of order λ with $f(0) = 1$, and let $\{a_1, a_2, \ldots\}$ be the zeros of f counted according to multiplicity and arranged so that $|a_1| \leq |a_2| \leq \cdots$. If an integer $p > \lambda - 1$ then prove that $\frac{d^p}{dz^p}\left(\frac{f'(z)}{f(z)}\right) = -p! \sum_{n=1}^{\infty} \frac{1}{(a_n - z)^{p+1}}$ for $z \neq a_1, a_2, \ldots$. **(15)**

OR

d) State and prove Hadamard's Factorization theorem. **(15)**

5. a) Show that $\square(z) - \square(u) = -\frac{\sigma(z-u)\sigma(z+u)}{\sigma(z)^2\sigma(u)^2}$.

OR

b) Prove that an elliptic function without poles is a constant. **(5)**

c) (i) Prove that a discrete module consists of either of zero alone, of the integral multiples nw of a single complex number $w \neq 0$ or of linear combinations $n_1 w_1 + n_2 w_2$ with integral coefficients of two numbers w_1, w_2 with non real ratio $\frac{w_2}{w_1}$.

(ii) Prove that $n_1 w_2 - n_2 w_1 = 2\pi i$ **(7+8)**

OR

d) Prove that $\square(z)$ is an elliptic function. **(15)**

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