LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

FIRST SEMESTER – **NOVEMBER 2019**

PMT 1501 – LINEAR ALGEBRA

Date: 30-10-2019 Dept. No. Max. : 100 Marks

Time: 01:00-04:00

Answer ALL the questions:

I. a) i) Let T be a linear operator on a finite dimensional space V and let c be a scalar. Prove that the following statements are equivalent:

1. c is a characteristic value of T.
2. The operator (T-cI) is singular.
3. det (T-cI) =0.

(OR) (5)

ii) Let $A = \begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix}$ be the matrix of a linear operator T defined on R^3 with respect to the standard ordered basis. Prove that A is diagonalizable.

b) i) State and prove Cayley Hamilton theorem,

(OR) (15)

ii) Let V be a finite dimensional vector space over F and T be a linear operator on V. Then prove that T is triangulable if and only if the minimal polynomial for T is a product of linear polynomials over F.

II. a) i) Let T be a linear operator on an n-dimensional vector space V. Let A be an n x n matrix. Then prove that characteristic and minimal polynomials for T have the same roots, except for multiplicities.

(OR) (5)

ii) Let W be an invariant subspace for T. The characteristic polynomial for the restriction operator T_w divides the characteristic polynomial for T. The minimal polynomial for T_w divides the minimal polynomial for T.

b. i) State and prove Primary Decomposition theorem.

(OR) **(15)**

ii) Let T be a linear operator on a finite dimensional space V. If T is diagonalizable and if $c_1,...,c_k$ are the distinct characteristic values of T, then prove that there exist linear operators $E_1,...,E_k$ on V such that

1. $T = c_1E_1 + ... + c_kE_k$ 2. $I = E_1 + ... + E_k$ 3. $E_iE_j = 0, i \neq j$ 4. Each E_i is a projection

5. The range of E_i is the characteristic space for T associated with c_i .

III. a. i) If $\mathcal{B}$ is an ordered basis for W_i, $1 \le i \le k$, then prove that the sequences $\mathcal{B} = (\mathcal{B}_1 \ldots \mathcal{B}_k)$ is an ordered basis for W.

(OR) **(5)**

ii) If W is an invariant subspace for T, then W is invariant under every polynomial in T. Prove that for each α in V, the conductor $S(\alpha;W)$ is an ideal in the polynomial algebra F[x].

b. i) State and prove Cyclic Decomposition Theorem. **(15)**

(OR)

ii) If W is T – admissible then prove that there exists a vector α ∈ V such that $W \cap Z(\sim;T) = \{0\}$. **(7)**

iii) Let T be a linear operator on a finite-dimensional vector space V. Let p and f be the minimal and characteristic polynomials for T, respectively. Then prove the following:

(1) p divides f.

(2) p and f have the same prime factors, except for multiplicities.

(3) $if\ p = f_1^{r_1} \ldots . f_k^{r_k}$ is the prime factorization of p, then

$f = f_1^{d_1} \ldots . f_k^{d_k}$ where d_i is the nullity of $f_i(T)^{r_i}$ divided by the degree of f_i. **(8)**

IV. a) (i) Let V be a finite dimensional inner product space and f be a linear function on V then prove that there is a unique vector β in V such that $f(\alpha) = (\alpha/\beta)\ \forall\ \alpha \in V$.

(OR) **(5)**

(ii) Let V be a complex vector space and 'f' a form on V such that $f(\alpha, \alpha)$ is real for every α. Then prove that f is Hermitian.

(b) (i) Let V and W be finite-dimensional inner product spaces over the same field, having the same dimension. If T is a linear transformation from V into W, then prove that the following are equivalent.

(i) T preserves inner products

(ii) T is and (inner product space) isomorphism.

(iii) T carries every orthonormal basis for V onto and orthonormal basis for W.

(iv) T carries some orthonormal basis for V onto an orthonormal basis for W. **(15)**

(OR)

(ii) Let V be a finite-dimensional inner product space, and let T be a self-adjoint linear operator on V. Then prove that there is an orthonormal basis for V, each vector of which is a characteristic vector for T. **(8)**

(iii) Let V be a finite-dimensional complex inner product space and let T be any linear operator on V. Then prove that there is an orthonormal basis for V in which the matrix of T is upper triangular. **(7)**

V. a) (i) Let V be the finite dimensional vector space over F. Let T be the linear operator on V. Then prove that T is diagonalizable iff the minimal polynomial for T has the form $f(x)=(x-c_1)(x-c_2)\ldots(x-c_k)$ where $c_{i,} c_{2,} \ldots c_k$ are distinct characteristic elements.

(OR) **(5)**

(b) Let V be a finite-dimensional vector space over a field of characteristic zero, and let f be a symmetric bilinear form on V. Then there is an ordered basis for V in which f is represented by a diagonal matrix.

b) (i) Let V be a finite-dimensional vector space over the field of complex numbers. Let f be a symmetric bilinear form on V which has rank r. Then prove that there is and ordered basis $\mathcal{B} = \{\beta_1 \ldots, \beta_n\}$ for V such that

(i) The matrix of f in the ordered basis $\mathcal{B}$ is diagonal;

(ii) $f(\ _j, \ _j) = \begin{cases} 1, & j = 1, \ldots, r \\ 0, & j > r. \end{cases}$

(15)

(OR)

(ii) Let V be an n-dimensional vector space over the field of real numbers, and let f be a symmetric bilinear form on V which has rank r. Then prove that there is an ordered basis $\{\beta_1, \beta_2, \ldots, \beta_n\}$ for V in which the matrix of f is diagonal and such that $f(\beta_j, \ _j) = \pm 1$, $j = 1, \ldots, r$. Furthermore, the number of basis vectors β_j for which $f(\beta_j, \ _j) = 1$ is independent of the choice of basis.

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